Designing for Development

Commentary on Saxe, de Kirby, Kang, Le, and Schneider

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As readers of this journal need not be reminded, it is not a trivial challenge to trace the development of an idea and the system of activity in which it originates and grows. In all complex systems, inevitable contingencies arise at multiple levels of organization and dynamics. When the systems involve people, levels of organization range from individuals to communities and often emerge and interact at different temporal scales. This problem is especially acute in educational endeavors where the goal is to position students to participate in a discipline's epistemology, which entails designing opportunities for students to have direct experience with how knowledge is created and revised in that discipline. This epistemic enterprise has multiple ancestors and paths of descent including Piaget's attention to genetic epistemology, Vygotsky's focus on tools and semiotic means to promote contact between everyday and disciplined interpretations of phenomena, and the progressive ideology and socially mediated views of knowledge set forth by G.H. Mead and J. Dewey – all at the turn of the last century.

Contemporary efforts to extend this tradition include design-based research which, as its name implies, generates accounts of disciplinary development in light of explicitly designed systems to support conceptual change. Design begins with analysis of the conceptual foundations of a disciplinary idea which almost always inspires rethinking what is currently taken as pedagogically appropriate. As they reflect on and unravel the threads of potential origins of disciplinary understanding, designers also must consider how to position learners to participate in the disciplinary practices (or approximations to the practices) in which these conceptual foundations were originated, sustained and even transformed. Hence, on the one hand, design researchers are obligated both to identify productive seeds of disciplined ideas [in the

research reported by Saxe, de Kirby, Kang, Le, & Schneider, this issue, the ancestral forms of mathematical activity that could be transformed into disciplined activity and understandings of integers and rational numbers] and to find ways to involve students in forms of activity that nurture these seeds into more recognizable disciplinary knowledge and practices. Thus, the design enterprise recounted in this article has two interrelated goals. First, Saxe et al. recast and extend the conceptual foundations of integer and rational number, which have typically been taught as extensions of arithmetic grounded in extensions of whole number collections of objects. Second, what is considered as mathematical activity is revised to include approximations to disciplinary practices that mathematicians have long found to be productive and useful across a range of ideas and realms of mathematical expression.

Seeds of Development

The first goal, deciding upon disciplinary seeds of rational number and integer, is managed in this design by appealing to a model of number as measured quantity. This model for number supplements the counting collections schemes that are assumed in this design to be already in place. Measure metaphorically grounds and extends these conceptions of number in two fundamental ways [Lakoff & Núñez, 2000]. The first is conceiving of a measured quantity, especially length, as constructed and its magnitude measured, by arranging and counting physical segments end to end. Lakoff and Núñez [2000], who wrote extensively about these properties of measure, apparently presume that it is evident that these physical segments should be identical and that they can be translated to tile a length. In fact, these features of unit are often not transparent to children [Lehrer, 2003]. But it does seem to be part of everyday experience to lay and count parts of bodies such as thumbs or feet to approximate lengths. Thus, even though students may or may not fully understand the (scientific) concept of translation, it seems likely that actions of displacement are available to serve as an activity scheme to support number as length. Because distances in the world are often not whole numbers of unit lengths, there is also a natural need to break units, to literally split them. Again, children often do not initially conceive of these breaks as involving equipartitioning. Nonetheless, it seems plausible that many will see the need for partial units, motivating the development of other kinds of numbers such as the rational number concepts targeted in this investigation. Lakoff and Núñez [2000] suggest a close counterpart to number as measure is number as location along a trajectory of motion, buttressing rational numbers as measured distances from a source [Lehrer & Pfaff, 2011]. For example, a teacher asks students to consider the distance traveled “if you were walking from 1 to 4 ...” [Saxe et al., this issue]. This travel metaphor grounds zero as a number that designates the origin of motion. For straight paths, vertical reflection symmetry about the origin suggests another species of number, the integers, another target of instruction here. Hence, the primitive or everyday ideas and actions suggested by Saxe et al. are firmly rooted in this alternative world of number as measure.

Yet one might wonder if these metaphoric extensions are temporary “fixes” that will later need to be discarded as learners develop and extend their conceptions of a discipline. They appear to provide entrée to powerful disciplinary ideas, but how much scope do these ideas encompass in fostering students’ subsequent mathemati-

Fig. 1. $a/b \times U$ length.

cal development? As it happens, this metaphor has the potential to be extensible in very productive ways [Lakoff & Núñez, 2000]. The metaphoric extension of physical segment to line segment suggests that numbers correspond to lengths even when their measures are not commensurate (that is, cannot be expressed with a common unit as an integer ratio). This realization provides a metaphoric grounding for irrational numbers as well as the integers and integer ratios targeted in this investigation. Because measures of length are ratios, additional entailments of numbers as measures include opportunities to incubate relational reasoning. Reasoning about relations such as equal, less, more, and times as long, instead of only measured values, grounds analysis of structure – a province of algebra. Moreover, when it is coupled with everyday experiences of symmetry and enlarging or reducing, the line segment-motion blend suggests other powerful relations to arithmetic [e.g., Lehrer & Slovin, 2014]. For example, in figure 1, the result of the operation of $a/b \times U$ length is generated by the visual experience of enlargement or reduction of a length and by the geometry of dilation and similarity. (This image of multiplication by a rational is supported by the technology of lined paper which provides equal interval and parallel lines for dilating any arbitrary length.¹) Hence, one can be confident that employing images of number as length and as motion along a path may set the stage for related forms of mathematical development (e.g., fractions as operators) and will not lead learners into a conceptual cul-de-sac. This kind of extensibility of underlying conceptual principles is an important consideration in design research that is oriented toward supporting development over the long term.

Despite this rich set of potential affordances in everyday activity, number as measured quantity also presents some challenges from a disciplined point of view. I mentioned previously that conceiving of a distance as a measured length entails understanding unit iteration as laying identical units end to end. This is translation, not merely displacement, as noted and exemplified here by Saxe et al. The measured quantity is the ratio of the spatial extent of the length to the spatial extent of the unit

¹ Personal communication, David W. Henderson, January 2011.

length. Thus, one should anticipate inverse change in the measure of a length when the unit length changes. In contrast, many children anticipate that longer units will lead to larger measures. The implication is that, although there are clearly opportunities to develop multiplicative comparisons among lengths by virtue of their measure with a common unit, these will need to be systematically developed during the course of instruction rather than regarded as mature everyday concepts to be deployed as needed.

Cultivating Seeds of Development by Engaging Students in Mathematical Practice

To resolve these and related challenges, Saxe et al. position students to participate in two genres of mathematical practice. The first is symbolizing mathematical activity by representing integers and rational numbers as locations on a number line. Symbolizing on a number line encourages the refinement of counting, displacement and splitting into their more disciplined counterparts of iteration and partitioning and helps to establish coordination between them so that partitions can be iterated and their measure established by counting. This coordinated activity generates rational numbers and, if reversible, integers. The second genre of mathematical practice highlighted here is definition. Definitions are the negotiated grounds for mathematical work and, as such, are distinct from axioms, which are taken for granted, or theorems, which can be proven. Defining is closely coupled with the development and revision of new concepts and ways of understanding. For example, Lakatos [1976] describes the historic relations between contested definitions of polyhedron and the genesis of topological understandings of shape. In the Saxe et al. research, students' definitions of integers, once negotiated collectively, then regulate and give meaning to individual activity. For example, students partition and iterate subdivided units to arrive at numbers such as $\frac{5}{3}$ and debate about whether numbers like these should be considered exemplars of rational numbers. Such contests extend the reach of rational number beyond the limited part-whole conceptions that students often develop. If one's conception of fraction is restricted to this familiar part-whole perspective, then $\frac{5}{3}$ is not sensibly a fraction because the number of parts exceeds the whole [Thompson & Saldanha, 2003]. Students are inducted into practices like these when the teacher invites them to take authority for definitions (and representations) and deliberately introduces errors to recreate contests about the common ground of a definition so that students must justify why the teacher's action or idea is not compatible with the definition. When students regularly engage in critiques of this sort, they begin to consider definitions as functional and not merely formal. At some times, efforts to destabilize a definition might be more confusing than mathematically productive, so there is also an unspoken element of teacher professional judgment that is nonetheless critical to the success of this enterprise. The analysis of definitional episodes developed by Saxe et al. documents the interplay between individual micro-genetic activity (what they call ostensive definitions) and the collective practice of negotiating a definition. These episodes clearly suggest that mathematical practices and concepts co-originate consistent with Hutchins's [2012] characterization of "concepts-in-practice." However, what is especially noteworthy is that these opportunities are systematically engineered and, hence, they dependably recur. Disciplin-

ary practices like definition are embedded within a more domain-general set of pedagogical practices such as partner participation structures. Hence, the disciplinary practices are not exclusively responsible for the development observed, but they do contribute substantially to the productive disciplinary engagement [Engle & Conant, 2002] evident in these episodes of definition and representation.

Managing Complexity

In light of the complexity and long record of classroom interaction, the analyses conducted point to a theoretically compelling yet pragmatically viable way to construct a history of learning. Recurrent forms of mathematical practice enabled the authors to create compelling trajectories of collective activity and learning over time while preserving the contributions of individual development. Such histories of learning will be needed if we are to make continued progress in designing for development. A hoped-for future innovation would make it possible to visualize individual and collective trajectories simultaneously. In the interim, this research stands as a landmark of productive simplifications of complex forms of interaction in a way that is tractable for both design and analysis. As in any good design, its elegance belies its difficulty.

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